Innovative Technique for Computing Shortest Length and/or Equal Tails Confidence Intervals in Reliability and Safety under Parametric Uncertainty

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Abstract. **A confidence interval is a range of values that provides the user with useful information about how accurately a statistic estimates a parameter. In the present paper, a new simple computation technique is proposed for simultaneous constructing and comparing confidence intervals of shortest-length and equal tails. This unified computation technique provides intervals in several situations that previously required separate analysis using more advanced methods and tables for numerical solutions. In contrast to the Bayesian approach, the proposed approach does not depend on the choice of priors and is a novelty in the theory of statistical decisions. It allows one to exclude nuisance parameters from the problem using the technique of invariant statistical embedding and averaging in terms of pivotal quantities (ISE & APQ) and quantile functions. It should be noted that the well-known classical approach to constructing confidence intervals of the shortest length considers at least three versions of possible solutions and is in need of information about the forms of probability distributions of pivotal quantities in order to determine an adequate version of the correct solution. The proposed technique does not need such information. It automatically recognizes an adequate version of the correct solution. To illustrate this technique, numerical examples are given.**

***Keywords*: parametric uncertainty, pivotal quantity, confidence intervals of shortest length and/or equal tails**

I. Introduction

To make a statistical inference in many problems under parametric uncertainty, the experimenter is interested in constructing a confidence interval that contains the true (unknown) value of the parameter (say, λ) with a given probability [1, 2]. If we are given a random sample $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ from a density $f_\lambda(y)$ and a pivotal quantity $V(S(\mathbf{Y}), \lambda)$ (which is developed from either maximum likelihood estimate or sufficient statistic $S(\mathbf{Y})$), whose distribution does not depend on λ, then confidence interval for a single unknown parameter λ is often derived by using a pivotal quantity $V(S(\mathbf{Y}), \lambda)$.

II. Classical Approach to Constructing Shortest Length Confidence Intervals

It is assumed that the length of the statistical confidence interval is given by

$$L(q_1, q_2 \mid S(\mathbf{Y})) \propto L(q_1, q_2) = \int_{q_1}^{q_2} \zeta(\tau) d\tau. \tag{1}$$

The length of the expected confidence interval is given by

$$E_\lambda\{L(q_1, q_2 \mid S(\mathbf{Y}))\} \propto L(q_1, q_2) = \int_{q_1}^{q_2} \zeta(\tau) d\tau. \tag{2}$$

In order to find the 100(1−α)% statistical (or expected) shortest-length confidence interval for λ , we should find a pair of decision variables q_1 and q_2 such that $L(q_1, q_2)$ is minimum.

A. *Problem Statement in Terms of the Pivot V and Decision Variables q_1 and q_2*

Minimize

$$L(q_1, q_2) = \int_{q_1}^{q_2} \zeta(\tau) d\tau \tag{3}$$

subject to

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$$\int_{q_1}^{q_2} q(v)dv = Q(q_2) - Q(q_1) = 1-\alpha. \quad (4)$$

B. Classical Analytical Approach to Solution of the Problem

Differentiating $L(q_1, q_2)$ with respect to q_1, we get

$$\frac{dL(q_1,q_2)}{dq_1} = \frac{d}{dq_1}\int_{q_1}^{q_2} \zeta(\tau)d\tau = \zeta(q_2)\frac{dq_2}{dq_1} - \zeta(q_1)\ . \quad (5)$$

From (4) we find the derivative of q_2 with respect to q_1 as follows:

$$\frac{d}{dq_1}\int_{q_1}^{q_2} q(v)dv = \frac{d}{dq_1}(1-\alpha) \quad (6)$$

that is

$$q(q_2)\frac{dq_2}{dq_1} - q(q_1) = 0. \quad (7)$$

Thus, we have

$$\frac{dq_2}{dq_1} = \frac{q(q_1)}{q(q_2)}. \quad (8)$$

Substituting this into (5), we obtain

$$\frac{dL(q_1,q_2)}{dq_1} = \zeta(q_2)\frac{q(q_1)}{q(q_2)} - \zeta(q_1). \quad (9)$$

Now consider the following three versions of possible decision making.

C. Version 1 of Possible Decision Making

$$\frac{dL(q_1,q_2)}{dq_1} = \zeta(q_2)\frac{q(q_1)}{q(q_2)} - \zeta(q_1) = 0. \quad (10)$$

Then the optimal analytical solution of the problem is given by

$$\frac{q(q_1)}{\zeta(q_1)} = \frac{q(q_2)}{\zeta(q_2)}. \quad (11)$$

D. Version 2 of Possible Decision Making

$$\frac{dL(q_1,q_2)}{dq_1} = \zeta(q_2)\frac{q(q_1)}{q(q_2)} - \zeta(q_1) < 0. \quad (12)$$

It follows from (4) that

$$1-\alpha < Q(q_2) \le 1. \quad (13)$$

Then the optimal analytical solution of the problem is given by

$$q_2 = \arg[Q(q_2) = 1], \quad q_1 = \arg[Q(q_1) = \alpha]. \quad (14)$$

E. Version 3 of Possible Decision Making

Let us assume that

$$Q(q_2) = 1 - \bar{Q}(q_2), \quad Q(q_1) = 1 - \bar{Q}(q_1). \quad (15)$$

It follows from (4) that

$$\int_{q_1}^{q_2} q(v)dv = Q(q_2) - Q(q_1) = \bar{Q}(q_1) - \bar{Q}(q_2) = 1-\alpha. \quad (16)$$

It follows from (16) that

$$\frac{d}{dq_1}\int_{q_1}^{q_2} q(v)dv = \frac{d}{dq_1}\left(\bar{Q}(q_1) - \bar{Q}(q_2)\right) = \frac{d}{dq_1}(1-\alpha) \quad (17)$$

that is

$$\bar{Q}'(q_1) - \bar{Q}'(q_2)\frac{dq_2}{dq_1} = 0. \quad (18)$$

Thus, we have

$$\frac{dq_2}{dq_1} = \frac{\bar{Q}'(q_1)}{\bar{Q}'(q_2)}. \quad (19)$$

Substituting this into (5), we obtain

$$\frac{dL(q_1,q_2)}{dq_1} = \zeta(q_2)\frac{\bar{Q}'(q_1)}{\bar{Q}'(q_2)} - \zeta(q_1) > 0. \quad (20)$$

It follows from (16) that

$$1-\alpha < \bar{Q}(q_1) \le 1, \quad (21)$$

Then the optimal analytical solution of the problem is given by

$$q_1 = \arg[\bar{Q}(q_1) = 1], \quad q_2 = \arg[\bar{Q}(q_2) = \alpha]. \quad (22)$$

III. New Simple Computation Technique of Constructing Confidence Intervals

It is assumed that the length of the confidence interval is given by (3). The proposed approach is based on the use of the numerical values of the quantile functions q_1 and q_2. In order to find the $100(1-\alpha)\%$ shortest-length confidence interval for λ, we should find a pair of numerical values of the quantile functions q_1 and q_2 such that $L(q_1, q_2)$ is minimum.

A. Problem Statement in Terms of the Decision Variable p (Probability) and Quantile Functions q_1 and q_2

Minimize

$$L^2(q_1,q_2) = \left(\int_{q_1}^{q_2} \zeta(\tau)d\tau\right)^2, \quad (23)$$

where the quantile function q_1 is given by

$$q_1 = Q^{-1}(p), \quad (24)$$

the quantile function q_2 is given by

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$$q_2 = Q^{-1}(1-\alpha+p), \tag{25}$$

subject to

$$0 \le p \le \alpha. \tag{26}$$

The decision variable to be determined is p (probability).

If $p = \alpha/2$ we have the confidence interval of equal tails.

B. New Simple Computation Method for Numerical Solution of the Problem

The optimal numerical values of the quantile functions q_1 and q_2, which minimize $L(q_1,q_2)$, can be obtain from (23)-(26) using computer software "Solver".

IV. Numerical Example 1

A. Classical Analytical Approach: The Problem Statement in Terms of the Pivot V and Decision Variables q_1 and q_2

Minimize

$$L(q_1,q_2) = \int_{q_1}^{q_2} \zeta(\tau)d\tau = \int_{q_1}^{q_2} \tau^{-2}d\tau = q_1^{-1} - q_2^{-1}. \tag{27}$$

subject to

$$\int_{q_1}^{q_2} q(v)dv = Q(q_2) - Q(q_1) = q_2^n - q_1^n = 1-\alpha, \tag{28}$$

where

$$q(v) = nv^{n-1}, \quad 0 < v < 1, \tag{29}$$

$$Q(u) = \int_0^u q(v)dv = u^n. \tag{30}$$

B. Analytical Solution of the Problem

$$\frac{dL(q_1,q_2)}{dq_1} = \zeta(q_2)\frac{d(q_2)}{d(q_1)} - \zeta(q_1) = \zeta(q_2)\frac{q(q_1)}{q(q_2)} - \zeta(q_1) =$$

$$= q_2^{-2}\frac{nq_1^{n-1}}{nq_2^{n-1}} - q_1^{-2} = \frac{q_1^{n-1}}{q_2^{n+1}} - \frac{1}{q_1^2} = \frac{q_1^{n+1} - q_2^{n+1}}{q_1^2 q_2^{n+1}} < 0, \tag{31}$$

where $q_2 > q_1$. It follows from (28) that

$$1-\alpha < Q(q_2) \le 1 \quad (\text{or } (1-\alpha)^{1/n} < q_2 \le 1). \tag{32}$$

For this example, the version 2 is an adequate version of possible decision making.

C. Optimal Analytical Results

It follows from the above that the minimum of $L(q_1,q_2)$ occurs at q_2 =1, $q_1 = \alpha^{1/n}$. Thus, the shortest length $(1-\alpha)$ - confidence interval is given by

$$L(q_1,q_2) = \frac{1}{\alpha^{1/n}} - 1. \tag{33}$$

If, say, n=3, α=0.1, then $L(q_1,q_2) = 1.154435$. Note that $L(q_1,q_2)$ goes to 0 as $n \to \infty$.

D. New Simple Computation Method: the Problem Statement in Terms of the Decision Variable p (Probability) and Quantile Functions q_1 and q_2

Minimize

$$L^2(q_1,q_2) = \left(\int_{q_1}^{q_2} \zeta(\tau)d\tau\right)^2 = \left(\int_{q_1}^{q_2} \tau^{-2}d\tau\right)^2 =$$

$$= \left(q_1^{-1} - q_2^{-1}\right)^2 = \left(\frac{1}{p^{1/n}} - \frac{1}{(1-\alpha+p)^{1/n}}\right)^2, \tag{34}$$

where the p–quantile function q_1 of V is given by

$$q_1 = Q^{-1}(p) = p^{1/n} \tag{35}$$

and the $(1-\alpha+p)$ -quantile function q_2 of V is given by

$$q_2 = Q^{-1}(1-\alpha+p) = (1-\alpha+p)^{1/n}, \tag{36}$$

subject to

$$0 \le p \le \alpha. \tag{37}$$

E. Numerical Solutions

The optimal numerical solution minimizing $L(q_1,q_2)$ can be obtained using the computer software "Solver". If, for example, n=3, α=0.1, then the optimal numerical solution is given by

$$p = 0.1, \quad q_1 = 0.464159, \quad q_2 = 1 \tag{38}$$

with the 100(1−α)% shortest-length confidence interval

$$L(q_1,q_2) = 1.154435. \tag{39}$$

The 100(1−α)% equal tails confidence interval is given by

$$L(q_1,q_2 \mid p = \alpha/2) = 1.697173 \tag{40}$$

with

$$p = 0.05, \quad q_1 = 0.368403, \quad q_2 = 0.983047572. \tag{41}$$

F. Inference

The proposed method correctly recognized the adequate version 2 of possible decision making and gave accurate numerical results.

G. Relative Efficiency

The relative efficiency of $L(q_1,q_2 \mid p = \alpha/2)$ as compared with $L(q_1,q_2)$ is given by

$$\text{rel.eff.}_L\left\{L(q_1,q_2 \mid p = a/2), L(q_1,q_2)\right\} =$$

$$= \frac{L(q_1,q_2)}{L(q_1,q_2 \mid p = a/2)} = \frac{1.154435}{1.697173} = 0.680211. \tag{42}$$

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V. Numerical Example 2

A. Classical Analytical Approach: The Problem Statement in Terms of the Pivot V and Decision Variables q_1 and q_2

Minimize

$$L(q_1,q_2)=\int_{q_1}^{q_2}\zeta(\tau)d\tau=\int_{q_1}^{q_2}\tau^{-2}d\tau=q_1^{-1}-q_2^{-1}. \quad (43)$$

subject to

$$\int_{q_1}^{q_2}q(v)dv=Q(q_2)-Q(q_1)=1-\alpha, \quad (44)$$

where

$$q(v)=\frac{1}{2^{n/2}\Gamma(n/2)}v^{n/2-1}\exp\left(-\frac{v}{2}\right),\quad v>0,\quad n>0, \quad (45)$$

$$Q(u)=\int_{-\infty}^{u}q(v)dv. \quad (46)$$

B. Analytical Solution of the Problem

It follows from (8) and (43) that

$$\frac{dL(q_1,q_2)}{dq_1}=\zeta(q_2)\frac{d(q_2)}{d(q_1)}-\zeta(q_1)=q_2^{-2}\frac{q(q_1)}{q(q_2)}-q_1^{-2}, \quad (47)$$

which vanishes if

$$q_2^{-2}\frac{q(q_1)}{q(q_2)}=q_1^{-2}. \quad (48)$$

For this example, the version 1 is an adequate version of possible decision making.

C. Optimal Analytical Results

It follows from (48) that the optimal solution is given by

$$q_1^2q(q_1)=q_2^2q(q_2). \quad (49)$$

Numerical results giving values of q_1 and q_2 to four significant places of decimals are available (see Tate and Klett [3]).

D. New Simple Computation Method: the Problem Statement in Terms of the Decision Variable p (Probability) and Quantile Functions q_1 and q_2

Minimize

$$L^2(q_1,q_2)=\left(\int_{q_1}^{q_2}\zeta(\tau)d\tau\right)^2=\left(\int_{q_1}^{q_2}\tau^{-2}d\tau\right)^2=\left(q_1^{-1}-q_2^{-1}\right)^2, \quad (50)$$

where the p–quantile function q_1 of V is given (via Excel software: CHISQ.INV (probability p, deg freedom n)) by

$$q_1=Q^{-1}(p), \quad (51)$$

and the $(1-\alpha+p)$-quantile function q_2 of V is given (via Excel software: CHISQ.INV (probability $1-\alpha+p$, deg freedom n)) by

$$q_2=Q^{-1}(1-\alpha+p), \quad (52)$$

subject to

$$0\le p<\alpha. \quad (53)$$

E. Numerical Solutions

The optimal numerical solution minimizing $L(q_1,q_2)$ can be obtained using the computer software "Solver". If, for example, n=3, α=0.1, then the optimal numerical solution is given by

$$p=0.099478,\quad q_1=0.58208,\quad q_2=17.63810464,$$
$$q(q_1)=0.227512,\quad q(q_2)=0.000248 \quad (54)$$

with

$$q_1^2q(q_1)=q_2^2q(q_2)=0.077. \quad (55)$$

and the 100(1−α)% shortest-length confidence interval

$$L(q_1,q_2)=1.661282. \quad (56)$$

The 100(1−α)% equal tails confidence interval is given by

$$L(q_1,q_2\mid p=\alpha/2)=2.714186 \quad (57)$$

with

$$p=0.05,\quad q_1=0.351846,\quad q_2=7.814728,$$
$$q(q_1)=0.198465,\quad q(q_2)=0.022409. \quad (58)$$

F. Inference

The proposed method correctly recognized the adequate version 1 of possible decision making and gave accurate numerical results.

G. Relative Efficiency

The relative efficiency of $L(q_1,q_2\mid p=\alpha/2)$ as compared with $L(q_1,q_2)$ is given by

$$\text{rel.eff.}_L\left\{L(q_1,q_2\mid p=a/2),L(q_1,q_2)\right\}=$$
$$=\frac{L(q_1,q_2)}{L(q_1,q_2\mid p=a/2)}=\frac{1.661282}{2.714186}=0.612074. \quad (59)$$

VI. Conclusion

The novel unified computation technique proposed in this paper represents the conceptually simple, efficient and useful method for constructing exact statistical (or expected) shortest-length or equal tails confidence intervals in terms of pivotal quantities and quantile functions. The exact confidence intervals with the shortest length or equal tails can be found easily and quickly. Applying the proposed novel unified computation technique, we are not in need to use the

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following: 1) analytical recognition and computational confirmation of adequate versions of possible solutions, 2) tables for numerical solutions, 3) more advanced methods and 4) special computer programs. For example, the special computer program for (49) is given below:

Minimize

$$z = [q_1^2 q(q_1) - q_2^2 q(q_2)]^2 \tag{60}$$

where the p–quantile function q_1 of V is given (via Excel software: CHISQ.INV (probability p, deg freedom n)) by

$$q_1 = Q^{-1}(p), \tag{61}$$

and the $(1-\alpha+p)$ -quantile function q_2 of V is given (via Excel software: CHISQ.INV (probability 1-α+p, deg freedom n)) by

$$q_2 = Q^{-1}(1-\alpha+p), \quad 62)$$

subject to

$$0 \le p < \alpha. \tag{63}$$

The optimal numerical solution minimizing z can be obtained using the computer software "Solver". If, for example n=3, α=0.1, then the optimal numerical solution is given by

$$p = 0.099478, \quad q_1 = 0.58208, \quad q_2 = 17.63813,$$

$$q(q_1) = 0.227512, \quad q(q_2) = 0.000248,$$

$$q_1^2 q(q_1) = q_2^2 q(q_2) = 0.077085. \tag{64}$$

with the 100(1−α)% shortest-length confidence interval

$$L(q_1, q_2) = q_1^{-1} - q_2^{-1} = 1.661282. \tag{65}$$

The main advantage of the proposed technique is that it includes only one decision variable (probability p) for making decisions under constraints. In other words, the two decision variables q_1 and q_2 are reduced to one decision variable (probability p). This technique greatly simplifies the problem of constructing shortest-length or equal tails confidence intervals for unknown parameters of various distributions and is a novelty in the theory of statistical decisions regarding confidence intervals. It allows one to exclude unknown (nuisance) parameters from the problem using the technique of invariant statistical embedding and averaging in terms of pivotal quantities (ISE & APQ) [4–14].

The unified computation method described in the paper is illustrated in detail for some selected cases. Applications of this method to construct shortest-length or equal tails confidence intervals for unknown parameters of log-location-scale or other probability distributions can follow directly.

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